Let $A \subseteq \mathbb{R}^p$ and f be a function defined over A. To define $\int_A f$, the integral of f over A, it is reasonable to require that

(a) the content (volume) of A is measurable, e.g. if $A = \prod_{j=1}^{p}[a_j, b_j]$ is a closed cell in $\mathbb{R}^p$, one can define its content to be $c(A) = \prod_{j=1}^{p}(b_j - a_j)$.

(b) the function f is summable, i.e. integrable, over A.

In the first part of this handout, we shall discuss how to define the integral of a function on cells in $\mathbb{R}^p$. In the second part of the handout, we shall extend the definition to a function on more general (measurable) sets in $\mathbb{R}^p$.

Part 1: Integrable Functions on Cells:

Definitions:

(a) K is called a **cell in $\mathbb{R}^p$ (or a $p-$cell, or a $p-$dimensional rectangle)** if $K = I_1 \times \cdots \times I_p$, where $\overline{I}_j = [a_j, b_j] \subset \mathbb{R}$ for $j = 1, \ldots, p$.

(b) **The ($p-$)content** $c(K)$ of K is defined to be $c(K) = (b_1 - a_1) \times \cdots \times (b_p - a_p) = \prod_{j=1}^{p}(b_j - a_j)$.

(c) **A set $Z \subset \mathbb{R}^p$ has $p-$content zero** if $\forall \, \epsilon > 0$, $\exists$ a finite set $\mathscr{C} = \{K_j\}_{j=1}^{m}$ of $p-$cells such that

(a) $Z \subset \bigcup_{j=1}^{m} K_j$,

(b) $\sum_{j=1}^{m} c(K_j) < \epsilon$.

Remark 1. Note that the definition implies that if K is a cell (not necessarily closed) in $\mathbb{R}^p$, then the boundary ∂K of K is a set of $p-$content zero.

Remark 2. Note that the definition of the content for a cell is well defined since it is easy to see that the following properties are satisfied.

(a) Let K be a cell in $\mathbb{R}^p$) and K is a finite disjoint union of cells in $\mathbb{R}^p$), i.e. $K = \bigcup_{i=1}^{l} K_i$, then

$$c(K) = \sum_{i=1}^{l} c(K_i).$$

(b) Let K_1, K_2 be cells in $\mathbb{R}^p$). Then $c(K_1 \cup K_2) = c(K_1 \setminus K_2) + c(K_1 \cap K_2) + c(K_2 \setminus K_1)$.

(c) Let $x \in \mathbb{R}^p$, K be a cell in $\mathbb{R}^p$) and $x + K = \{x + z \mid z \in K\}$. Then $x + K$ is a cell in $\mathbb{R}^p$) with $c(x + K) = c(K)$, i.e. the definition of content for cells is invariant under translations.

Remark 3. By taking $\epsilon/2 > 0$, if it is necessary, one may also assume that $Z \subset \text{Int}\left(\bigcup_{j=1}^{m} K_j\right)$.

Example (1)**.** Let $Z = \{x_j \in \mathbb{R} \mid \lim_{j\to\infty} x_j = x\}$, a (0-dim'l) subset of $\mathbb{R}$. Then $(1 - \text{dim'l})$ $c(Z) = 0$ since $\forall \epsilon > 0$, $\exists$ a 1-d cell K_x such that $x \in \text{Int}(K_x)$, $c(K_x) < \epsilon/2$, and $x_j \in K_x \, \forall j \geq L$. For each $j = 1, \ldots, L-1$, let K_j be a 1-d cell such that $x_j \in K_j$, and $c(K_j) \leq \epsilon/(2L)$.

Example (2)**.** Let $Z = \mathbb{Q} \cap [0, 1]$, a (0-dim'l) subset of $\mathbb{R}$. Then (1-dim'l) $c(Z) \neq 0$ since any finite collection $\mathscr{C} = \{K_1, \ldots, K_m\}$ of $1-$dimensional cells that satisfies (a) will have $\sum_{j=1}^{m} c(K_j) \geq 1$.

Example (3)**.** Let $Z = \{(x,y) \mid |x| + |y| = 1\}$, a (1-dim'l) subset of $\mathbb{R}^2$. Then the (2-d) content $c(Z) = 0$.

Definition (4). A collection of sets $\mathscr{C} = \{K_j\}_{j=1}^m$ in $\mathbb{R}^n$ is called **a partition of a set** K in $\mathbb{R}^n$ if

(a) $\bigcup_{i=1}^{m} K_i = K$, and

(b) $\mathrm{Int}K_i \cap \mathrm{Int}K_j = \emptyset$ holds for each $1 \leq i \neq j \leq m$.

Example. Let $K = [a_1, b_1] \times \cdots \times [a_n, b_n] = I_1 \times \cdots \times I_n$, and $P_j = \{[x_i^j, x_{i+1}^j] \mid a_j = x_0^j < \cdots < x_{m(j)}^j = b_j\}$, for $j = 1, \ldots, n$, be a partition of I_j into $m(j)$ (a finite number of) closed cells in $\mathbb{R}$. Then the set $P = \{\prod_{j=1}^n [x_{i_j}^j, x_{i_j+1}^j] \mid 0 \leq i_j \leq m(j) - 1\}$ (induced by P_j's) **partitions** K **into** $m(1) \times \cdots \times m(n)$ **(finite number of) parallel closed n-cells.**

Example. Let $K = [0,1] \times [2,4] \subset \mathbb{R}^2$, and for each $n = 1, 2, \ldots$, let $P_n = \{[\frac{i-1}{n}, \frac{i}{n}] \times [2 + \frac{2(i-1)}{n}, 2 + \frac{2i}{n}] \mid 1 \leq i \leq n\}$ be a partition of K that divides each side of K into n equal length subintervals. One can define the norm of a partition P_n to be the $\|P_n\| = \max_{K_j \in P_n} \mathrm{diam}(K_j)$. In this example, the norm of the partition is $\|P_n\| = \frac{\sqrt{5}}{n}$.

Definition(5). Let $P = \{I_i\}_{i=1}^l$ and $Q = \{K_j\}_{j=1}^m$ be partitions of (an n-cell) K. We say that P **is a refinement of** Q**, denoted** $Q \subset P$**, if each cell in** P **is contained in some cell in** Q, i.e. for each $I_i \in P\ \exists\, K_j \in Q$ such that $I_i \subset K_j$.

Note that if P, Q are partitions of K, then $P \cap Q$ **is a (common) refinement of** P **and** Q, and, **in general,** $P \cup Q$ **is Not a partition,**

Example. Let $K = [0,1] \times [2,4] \subset \mathbb{R}^2$, and for each $n = 1, 2, \ldots$, let $P_n = \{[\frac{i-1}{n}, \frac{i}{n}] \times [2 + \frac{2(i-1)}{n}, 2 + \frac{2i}{n}] \mid 1 \leq i \leq n\}$. Then $I = [0, 1/2] \times [2, 3] \in P_2 \subset P_2 \cup P_3$, $J = [0, 1/3] \times [2, 8/3] \in P_3 \subset P_2 \cup P_3$, but $(0, 1/2) \times (2, 3) \cap (0, 1/3) \times (2, 8/3) \neq \emptyset$. Therefore, $P_2 \cup P_3$ is not a partition of K.

Definition (6)**.** Let f be **a bounded function defined on a closed** $n-$**cell** K **with values in** $\mathbb{R}$. **A Riemann sum** $S_P(f, K)$ **corresponding to a partition** $P = \{K_j\}_{j=1}^m$ **of** K **is given by** $S_P(f, K) = \sum_{i=1}^{m} f(x_i) c(K_i)$, **where** x_i **is any point in** K_i, **and** $c(K_i)$ **denotes the (n-dim'l) content of** K_i.

Remark. Note that $\sum_{i=1}^{m} m_i c(K_i) = L_P(f) \leq S_P(f) = S_P(f, K) \leq U_P(f) = \sum_{i=1}^{m} M_i c(K_i)$, where $m_i = \inf_{K_i} f \leq f(x_i) \leq M_i = \sup_{K_i} f$, and $L_P(f)$, and $U_P(f)$ are called the **lower sum and upper sum, respectively, of** f **with respect to the partition** P **of** K.

Remark. (Monotonicity of lower and upper sums) If P, Q are partitions of K, and $P \subset Q$ i.e. Q is finer than P, then we have

$$L_P(f) \leq L_Q(f) \leq S_Q(f) \leq U_Q(f) \leq U_P(f).$$

Since the set $\{L_P(f) \mid P \,\text{is a partition of}\, K\}$ is nonempty, and bounded from above by $\big(\sup_K f\big) c(K)$, the $L(f, K) = \sup_P L_P(f) = \sup\{L_P(f) \mid P \,\text{is a partition of}\, K\}$ exists.

Analogously, the $U(f, K) = \inf_P U_P(f) = \inf\{U_P(f) \mid P \,\text{is a partition of}\, K\}$ exists.

If $\{P_k\}$ be any sequence of partitions of K such that $P_j \subset P_{j+1}$ for each $j = 1, 2 \ldots$, and $\|P_j\| \geq \|P_{j+1}\| \to 0$, then $\lim_{\|P_j\| \to 0} L_{P_j}(f) = L(f, K)$, and $\lim_{\|P_j\| \to 0} U_{P_j}(f) = U(f, K)$.

Definition (of integrability on cells). A bounded function f is called **Riemann integrable on** K if $L(f, K) = U(f, K)$ and this common value, denoted $\int_K f$, is called the **(Riemann) integral of** f **on** K.

Remark. f is integrable on K if and only if there exists a unique number L such that for each partition P of K we have $L_P(f) \leq L \leq U_P(f)$

Proof: ($\Rightarrow$) Since $L_P(f) \leq L(f,K) = U(f,K) \leq U_P(f)$ holds for each partition P of K, by setting $L = L(f,K)$, the inequality $L_P(f) \leq L \leq U_P(f)$ holds for each partition P of K. Suppose that L_1 is a number such that the inequality $L_P(f) \leq L_1 \leq U_P(f)$ holds for each partition P of K. **Then L_1 is an upper** (resp. a lower) **bound of the set $\{L_P(f) \mid P \text{ is a partition of } K\}$** (resp. $\{U_P(f) \mid P \text{ is a partition of } K\}$) **which implies that** $L = L(f,K) \leq L_1$ (resp. $L_1 \leq U(f,K) = L$)). Hence, $L_1 = L$ is the unique number such that the inequality $L_P(f) \leq L \leq U_P(f)$ holds for each partition P of K.

($\Leftarrow$) In order to show that $L(f,K) = U(f,K)$, we show that $L(f,K) = L$, and $U(f,K) = L$. Since the inequality $L_P(f) \leq L \leq U_P(f)$ holds for each partition P of K, we have $L(f,K) = \sup_P L_P(f) \leq L \leq \inf_P U_P(f) = U(f,K)$. **Suppose that** $L(f,K) < L$ (resp. $L < U(f,K)$), **then** $\exists\, \epsilon_0 > 0$ **such that** $L(f,K) < L - \epsilon_0$ (resp. $L + \epsilon_0 < U(f,K)$). **Thus, for each partition P of K, we have** $L_P(f) \leq L(f,K) < L - \epsilon_0 < L \leq U_P(f)$ (resp. $L_P(f) \leq L < L + \epsilon_0 < U(f,K) \leq U_P(f)$) **which contradicts to the uniqueness of L. This implies that** $L(f,K) = L$ (resp. $U(f,K) = L$), and $L(f,K) = L = U(f,K)$, i.e. f is integrable on K.

Criterion of integrability: Let f be a bounded function defined on K. Then the following are equivalent.

(1) f is integrable on K, i.e. $L(f,K) = U(f,K)$, with integral $L = \int_K f = L(f,K)$

(2) (Riemann Criterion for Integrability) $\forall\, \epsilon > 0$, $\exists$ partition P_ϵ, of K, such that if $P$is a refinement of P_ϵ, then $|U_P(f) - L_P(f)| < \epsilon$.

(3) (Cauchy Criterion for Integrability) $\forall\, \epsilon > 0$, $\exists$ partition P_ϵ, of K, such that if P and Q are any refinements of P_ϵ, and $S_P(f,K)$ and $S_Q(f,K)$ are any corresponding Riemann sums, then $|S_P(f,K) - S_Q(f,K)| < \epsilon$.

(4) $\forall\, \epsilon > 0$, $\exists$ partition P_ϵ, of K, such that if P is any refinement of P_ϵ, and $S_P(f,K)$ is any corresponding Riemann sum, then $|S_P(f,K) - L| < \epsilon$.

Proof Since $L_P(f) \leq L(f,K) \leq U(f,K) \leq U_P(f)$, (by drawing a picture) one notes that
$|U(f,K) - L(f,K)| \leq^{(*)} |U_P(f) - L_P(f)| \leq^{(\dagger)} |U_P(f) - L| + |L_P(f) - L|$.

$\boxed{\boxed{(1) \Rightarrow (2):}}$ Given $\epsilon > 0$, since $L(f,K) = U(f,K)$ (and by the definitions that $L(f,K)$ being the smallest number that satisfies $L(f,K) \geq L_P(f)$, and $U(f,K)$ being the largest number that satisfies $U(f,K) \leq U_P(f)$ for all P), there exists a partition P_ϵ such that

$$L(f,K) - \epsilon/2 < L_{P_\epsilon}(f) \leq L(f,K),$$

and

$$L(f,K) = U(f,K) \leq U_{P_\epsilon}(f) < U(f,K) + \epsilon/2 = L(f,K) + \epsilon/2.$$

Thus, if P is any refinement of P_ϵ, then

$$L(f,K) - \epsilon/2 < L_{P_\epsilon}(f) \leq L_P(f) \leq U_P(f) \leq U_{P_\epsilon}(f) < L(f,K) + \epsilon/2.$$

Setting $L = L(f,K)$ in the (second) inequality ($\dagger$), we get that

$$|U_P(f) - L_P(f)| \leq |U_P(f) - L| + |L_P(f) - L| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus, the conclusion of (2) holds.

$\boxed{\boxed{(2) \Rightarrow (1):}}$ For each $\epsilon > 0$, since the (first) inequality ($*$), and (2) hold, there exists a partition P_ϵ such that if P is any refinement of P_ϵ, then

$$|U(f,K) - L(f,K)| \leq |U_P(f) - L_P(f)| < \epsilon.$$

Letting $\epsilon \to 0$, we get

$$0 \leq \lim_{\epsilon \to 0} |U(f,K) - L(f,K)| \leq \lim_{\epsilon \to 0} \epsilon = 0 \Rightarrow U(f,K) = L(f,K)$$

and f is integrable.

$\boxed{\boxed{(2) \Leftrightarrow (3):}}$ For any refinements P, Q of P_ϵ, we have

$$L_{P_\epsilon}(f) \le L_P(f) \le S_P(f,K) \le U_P(f) \le U_{P_\epsilon}(f)$$
$$L_{P_\epsilon}(f) \le L_Q(f) \le S_Q(f,K) \le U_Q(f) \le U_{P_\epsilon}(f)$$

Thus, we have

$$|S_P(f,K) - S_Q(f,K)| \le |U_{P_\epsilon}(f) - L_{P_\epsilon}(f)|$$

and (2) $\Rightarrow$ (3).
Conversely, for any refinements P, Q of P_ϵ, if $|S_P(f,K) - S_Q(f,K)| < \epsilon/2$ then, since

$$|U_P(f) - L_Q(f)| = \sup |S_P(f,K) - S_Q(f,K)|,$$

where the supremum is taken on all possible Riemann sum $S_P(f,K)$ and $S_Q(f,K)$ corresponding to the given (fixed) partitions P and Q, respectively, we have

$$|U_P(f) - L_Q(f)| \le \sup |S_P(f,K) - S_Q(f,K)| \le \epsilon/2$$

and (3) $\Rightarrow$ (2).

$\boxed{\boxed{(3) \Leftrightarrow (4):}}$ Let $\{Q_j\}$ be a sequence of refinements of P_ϵ such that $Q_j \subset Q_{j+1}$ and $\lim_{j\to\infty} \|Q_j\| = 0$. then

$$|S_P(f,K) - L| = \lim_{j\to\infty} |S_P(f,K) - S_{Q_j}(f,K)| \le \epsilon$$

and (3) $\Rightarrow$ (4). Conversely, since

$$|S_P(f,K) - S_Q(f,K)| \le |S_P(f,K) - L| + |S_Q(f,K) - L|$$

holds, we have (4) $\Rightarrow$ (3).

Example (1). For $a < b$, let $f(x) = \begin{cases} a & \text{if } x \in \mathbb{Q} \cap [0,1], \\ b & \text{if } x \in [0,1] \setminus \mathbb{Q}. \end{cases}$
Then f is not continuous at each $x \in [a,b]$ and f is not integrable on $[0,1]$ since $L_P(f,[0,1]) = a \ne b = U_P(f,[a,b])$ for any partition P of $[a,b]$.

Example (2). Let $f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q} \cap [0,1], \text{ where } m, n \in \mathbb{N} = \{1,2,\ldots\} \text{ and } \gcd(m,n) = 1, \\ 1 & \text{if } x = 0, \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q}. \end{cases}$
Then f is integrable on $[0,1]$ and f is continuous at every irrational and discontinuous at every rational.

Observations. (1) There are finitely many rational numbers $\frac{p}{q} \in [0,1]$ such that $q < n$. In fact, for fixed $q < n$, the number of $\frac{p}{q} \in [0,1]$ is at most $q+1$, which is at most n. Moreover, there are less than n positive q such that $q < n$. Thus, the set $A_n = \{\frac{p}{q} \in [0,1] : q < n\}$ contains no more than n^2 element and note that if $\frac{m}{n} \in (0,1)$ is in lowest terms (m and n have no common factors except one), then $\min\{|\frac{p}{q} - \frac{m}{n}| : \frac{p}{q} \in A_n\} > \frac{1}{n^2}$.
(2) Fix $2 \le n \in \mathbb{N}$, and let $\hat{P} = \{\frac{p}{q} \pm \frac{1}{n^3} : q < n,\, q \in \mathbb{N},\, p = 0, 1, \ldots, q-1\}$. Since the set $\hat{P}$ is finite, it yields a partition $P = \big(\hat{P} \cap [0,1]\big) \cup \{1\}$ of $[0,1]$. Define a step function

$$s_n(x) = \begin{cases} 1 & \text{if } \exists\, q \in \mathbb{N},\, p \in \{0,1,\ldots,q-1\} \text{ with } q < n \text{ such that } \frac{p}{q} - \frac{1}{n^3} < x < \frac{p}{q} + \frac{1}{n^3}, \\ 0 & \text{if there do not exist such } p \text{ and } q. \end{cases}$$

Also, define $f_n(x) = \begin{cases} f(x) & \text{if } \exists\, q \in \mathbb{N},\, p \in \{0,1,\ldots,q-1\} \text{ with } q < n \text{ such that } \frac{p}{q} - \frac{1}{n^3} < x < \frac{p}{q} + \frac{1}{n^3}, \\ 0 & \text{if there do not exist such } p \text{ and } q. \end{cases}$

For each $n \geq 2$, since A_n contains no more than n^2 elements, there exist no more than n^2 intervals $(\frac{p}{q} - \frac{1}{n^3}, \frac{p}{q} + \frac{1}{n^3})$ in the interval $[0,1]$. Thus, we have $0 \leq U(f_n, [0,1]) \leq U_P(f_n, [0,1]) \leq U_P(s_n, [0,1]) = \int_0^1 s_n(x) \leq n^2\big(\frac{2}{n^3}\big) = \frac{2}{n}$ for all $n \geq 2$. By letting n go to infinity, we get $0 = U(f, [0,1]) \geq L(f, [0,1]) = 0$. This proves that f is integrable on $[0,1]$.
(3) Let $x = \frac{m}{n}$ be a rational number in lowest terms. Note that if $y \in [0,1]$ satisfying that $|y - x| < \frac{1}{4n^2} = \frac{1}{(2n)^2}$ then either $y \in [0,1] \setminus \mathbb{Q}$ or $y = \frac{p}{q}$ with $q < 2n$ (since any $\frac{p}{q} \in A_{2n}$ will have $|x - \frac{p}{q}| \geq \frac{1}{(2n)^2}$). Assume f is continuous at x. Given $\epsilon = \frac{1}{2n}$, there exists $\delta > 0$ such that if $y \in [0,1]$ and $|y - x| < \delta$ then $|f(x) - f(y)| < \epsilon$. Let $d = \min\{\delta, \frac{1}{(2n)^2}\}$ and note that if $y \in [0,1]$ and $|y - x| < d$ then $|f(x) - f(y)| > \frac{1}{2n}$ (regardless that y is rational or irrational). This contradicts our assumption.
(4) Let $\alpha \in [0,1] \setminus \mathbb{Q}$. Given $\epsilon > 0$, choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. Since there are finitely many rational numbers $\frac{p}{q} \in [0,1]$ such that $q < n$, the minimum distance δ between $\frac{p}{q}$ and α for $q < n$, i.e. $\delta = \min\{|\alpha - \frac{p}{q}| : q < n\}$, exists and it is positive since α is irrational. If $|x - \alpha| < \delta$, then either $x \in [0,1] \setminus \mathbb{Q}$, or $x = \frac{p}{q}$ with p and q having no common factors except one and $q \geq n$, since $|x - \alpha| < \delta$. Thus, we have $|f(x) - f(\alpha)| = \begin{cases} 0 < \epsilon & \text{if } x \in (\alpha - \delta, \alpha + \delta) \setminus \mathbb{Q}, \\ |f(\frac{p}{q})| = \frac{1}{q} \leq \frac{1}{n} < \epsilon & \text{if } x = \frac{p}{q} \in (\alpha - \delta, \alpha + \delta). \end{cases}$

Some basic properties of integrable functions on cells:
(1) Suppose that K, K_1, K_2 are closed $n-$cells such that $K = K_1 \cup K_2$, and $\text{Int}(K_1) \cap \text{Int}(K_2) = \emptyset$. If f is integrable on K, then f is integrable on K_1, and K_2, and $\int_K f = \int_{K_1} f + \int_{K_2} f$.
Proof. Given $\epsilon > 0$, since f is integrable, there is a partition P_ϵ of K such that if P is any refinement of P_ϵ, then $|U_P(f,K) - L_P(f,K)| < \epsilon$. For $i = 1, 2$, let $P_{\epsilon,i} = P_\epsilon \cap K_i$, then $P_{\epsilon,i}$ is a partition of K_i, and $P_{\epsilon,1} \cup P_{\epsilon,2}$ is a refinement of P_ϵ. Thus, we have

$$\begin{aligned} \epsilon &> U_{P_{\epsilon,1} \cup P_{\epsilon,2}}(f, K_1 \cup K_2) - L_{P_{\epsilon,1} \cup P_{\epsilon,2}}(f, K_1 \cup K_2) \\ &= U_{P_{\epsilon,1}}(f, K_1) - L_{P_{\epsilon,1}}(f, K_1) + U_{P_{\epsilon,2}}(f, K_2) - L_{P_{\epsilon,2}}(f, K_2) \\ &\geq U_{P_{\epsilon,i}}(f, K_i) - L_{P_{\epsilon,i}}(f, K_i) \\ &\geq 0 \end{aligned}$$

Thus, for each $i = 1, 2$,

$$\epsilon > U_{P_{\epsilon,i}}(f, K_i) - L_{P_{\epsilon,i}}(f, K_i) \geq 0$$

and if P_i is any refinement of $P_{\epsilon,i}$, then

$$\epsilon > U_{P_{\epsilon,i}}(f, K_i) - L_{P_{\epsilon,i}}(f, K_i) \geq U_{P_i}(f, K_i) - L_{P_i}(f, K_i) \geq 0.$$

This implies that f is Riemann integrable on K_i and $L_i = \int_{K_i} f$ exists, for $i = 1, 2$, and

$$\begin{aligned} L = L(f,K) &= \sup\{L_P(f) \mid P \,\text{is any refinement of}\, P_\epsilon\} \\ &\leq \sup\{L_{P_1}(f) + L_{P_2}(f) \mid P_i \,\text{is any refinement of}\, P_{\epsilon,i}\, i = 1,2\} \\ &\leq L_1 + L_2 \\ &\leq \inf\{U_{P_1}(f) + U_{P_2}(f) \mid P_i \,\text{is any refinement of}\, P_{\epsilon,i}\, i = 1,2\} \\ &= \inf\{U_P(f) \mid P \,\text{is any refinement of}\, P_{\epsilon,1} \cup P_{\epsilon,2}\} \\ &= U(f,K) = L \end{aligned}$$

Thus, we have $L = L_1 + L_2$.
(2) If f and g are integrable on K, then, for any $c \in \mathbb{R}$, $cf + g$ is integrable on K, and $\int_K (cf + g) = c \int_K f + \int_K g$.
Proof. Given $\epsilon > 0$, since f, g are integrable on K, there exists a partition P_ϵ of K such that if P is any refinement of P_ϵ then

$$|U_P(f) - L_P(f)| < \epsilon/2(1 + |c|)$$

and

$$|U_P(g) - L_P(g)| < \epsilon/2.$$

Thus, we have

$$|U_P(cf+g) - L_P(cf+g)| \le |c||U_P(f) - L_P(f)| + |U_P(g) - L_P(g)| < |c|\epsilon/2(1+|c|) + \epsilon/2 < \epsilon$$

which implies that $cf+g$ is integrable on K. Let $\{P_j\}$ be a sequence of partitions of K satisfying that $P_j \subset P_{j+1}$ for all $j = 1, 2, \ldots$, and $\lim_{j\to\infty} \|P_j\| = 0$. Since

$$\int_K f = \lim_{j\to\infty} S_{P_j}(f, K) \text{ and } \int_K g = \lim_{j\to\infty} S_{P_j}(g, K),$$

we have

$$\begin{aligned} c\int_K f + \int_K g &= c\lim_{j\to\infty} S_{P_j}(f,K) + \lim_{j\to\infty} S_{P_j}(g,K) \\ &= \lim_{j\to\infty} S_{P_j}(cf,K) + \lim_{j\to\infty} S_{P_j}(g,K) \\ &= \lim_{j\to\infty} S_{P_j}(cf+g,K) \\ &= \int_K (cf+g). \end{aligned}$$

(3) Suppose that f and g are integrable on K. If $f(x) \le g(x)$ for each $x \in K$, then $\int_K f \le \int_K g$.
Proof. Since $-f+g \ge 0$ on K and, by (2), it is integrable on K, we have

$$0 \le L_P(-f+g) \le \int_K (-f+g) = -\int_K f + \int_K g,$$

where P is any partition of K. Since $\int_K f \in \mathbb{R}$, by adding $\int_K f$ on both sides of the inequality, we get $\int_K f \le \int_K g$.
(4) If $f$is integrable on K, then $|f|$ is integrable on K, and $|\int_K f| \le \int_K |f|$.
Proof. Given $\epsilon > 0$, since f is integrable on K, there exists a partition P_ϵ of K such that if $P = \{K_j\}_{j=1}^m$ is any refinement of P_ϵ then

$$|U_P(|f|) - L_P(|f|)| = |\sum_{i=1}^m \big(\sup_{K_i} |f| - \inf_{K_i} |f|\big)c(K_i)| \le |\sum_{i=1}^m \big(\sup_{K_i} f - \inf_{K_i} f\big)c(K_i)| = |U_P(f) - L_P(f)| < \epsilon.$$

Thus, $|f|$ is integrable (by Riemann's Criterion for integrability).
Since $\pm f$, $|f|$ are integrable, and $\pm f \le |f|$ on K, we have $\pm\int_K f \le \int_K |f| \Rightarrow |\int_K f| \le \int_K |f|$.
Examples of integrability.
(1) Let $Z \subset \mathbb{R}^n$ have (n-)content zero, and f be a bounded function defined on Z. Then f is integrable on Z, and $\int_Z f = 0$.
Proof. For each $\epsilon > 0$, since Z has content zero, there exists a collection of cells $\{K_i\}_{i=1}^m$ such that $Z \subset \cup_{i=1}^m K_i = K$, and $\sum_{i=1}^m c(K_i) < \epsilon/2\big(\sup_Z |f| + 1\big)$. Define

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in Z, \\ 0 & \text{if } x \in K \setminus Z. \end{cases}$$

Then,

$$|\bar{f}|(x) = \begin{cases} |f|(x) \ge 0 & \text{if } x \in Z, \\ 0 & \text{if } x \in K \setminus Z, \end{cases}$$

and note that

$$\begin{aligned}
|\sup_{K_j\cap Z} f - \inf_{K_j\cap Z} f| &= |\sup_{K_j\cap Z} \bar f - \inf_{K_j\cap Z} \bar f| \\
&\le |\sup_{K_j} \bar f - \inf_{K_j} \bar f| \\
&\le |\sup_{K_j} |\bar f| - \inf_{K_j} \big(-|\bar f|\big)| \\
&= |\sup_{K_j} |\bar f| - \big(-\sup_{K_j} |\bar f|\big)| \\
&= 2\sup_{K_j} |\bar f| = 2\sup_{K_j\cap Z} |f| \\
&\le 2M.
\end{aligned}$$

Thus, if P is any partition of $K = \cup_{i=1}^m K_i$, we have

$$|U_P(f,Z) - L_P(f,Z)| \le 2\sup_Z |f| \sum_{i=1}^m c(K_i) < \epsilon$$

which implies that f is integrable on Z with $\int_Z f = 0$.
(2) Let I be a closed interval in $\mathbb{R}$, and f be a bounded and monotonic function defined on $I = [a,b]$ Then f is integrable on I.
Proof. Since f is bounded on I, $L(f,I) = \sup_P L_P(f)$ and $U(f,I) = \inf_P U_P(f)$ exist.
Let P_n be the partition that divides I into 2^n equal length subintervals. Thus,

$$\lim_{n\to\infty} L_{P_n}(f) = L(f,I) \text{ and } \lim_{n\to\infty} U_{P_n}(f) = U(f,I).$$

Since

$$\lim_{n\to\infty} |U_{P_n}(f) - L_{P_n}(f)| = \lim_{n\to\infty} \big(f(b)-f(a)\big)\big(b-a\big)/2^n = 0,$$

we get $L(f,I) = U(f,I)$, i.e. f is integrable on I.
(3) Let K be a closed $n-$cell, and f be a continuous function on K. Then f is integrable on K.
Proof. Since f is continuous on (compact set) K, f is uniformly continuous on K.
Hence, for any given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\text{if } x,y \in K \text{ and } \|x-y\| < \delta \text{ then } |f(x)-f(y)| < \epsilon/\big(c(K)+1\big).$$

Let P_ϵ be a partition of K such that $\|P_\epsilon\| = \max_{K_j\in P_\epsilon} \text{diam}(K_j) = \max_{K_j\in P_\epsilon} \sup\{\|x-y\| : x,y\in K_j\} < \delta$.
If P is any refinement of P_ϵ then $|U_P(f) - L_P(f)| < \epsilon c(K)/\big(c(K)+1\big) < \epsilon$.
Therefore, f is integrable on K.
(4) Let K be a closed $n-$cell, and f be a bounded function defined on K. If there exists a (n-)content zero subset $Z \subset K$, such that f is continuous on $K \setminus Z$, i.e. f is continuous everywhere on K except at a content zero subset Z of K, then f is integrable on K.
Proof. For each $\epsilon > 0$, since Z has content zero, there exists a collection of cells $\{I_i\}_{i=1}^l$ such that

$$Z \subset \text{Int}\big(\bigcup_{i=1}^l I_i\big),\ \text{Int}I_i \cap \text{Int}I_j = \emptyset,\ \text{and } \sum_{i=1}^l c(I_i) < \epsilon/4\big(\sup_K |f| + 1\big).$$

Since $K \setminus \text{Int}\big(\bigcup_{i=1}^l I_i\big)$ is compact, f is uniformly continuous there.
Thus, for the given $\epsilon > 0$, there exists a $\delta > 0$ such that if

$$x,y \in K_j \setminus \text{Int}\big(\bigcup_{i=1}^l I_i\big),\ \text{then } |f(x)-f(y)| < \epsilon/2\big(c(K)+1\big).$$

Let $P_\epsilon = \{K_j\}_{j=1}^m$ be a partition of K such that $\{I_i \cap K\}_{i=1}^l \subset P_\epsilon$ and $\|P_\epsilon\| = \max_{1\le j\le m} \sup_{x,y\in K_j} \|x-y\| < \delta$. If P is any refinement of P_ϵ then, by using (1) and (3), we have

$$\begin{aligned}
|U_P(f) - L_P(f)| &\le \sum_{i=1}^{l} \big(\sup_{Z\cap I_i} f - \inf_{Z\cap I_i} f\big)c(I_i) + |U_P(f, K\setminus \mathrm{Int}\big(\cup_{i=1}^l I_i\big)) - L_P(f, K\setminus \mathrm{Int}\big(\cup_{i=1}^l I_i\big))| \\
&\le 2\sup_Z |f| \sum_{i=1}^{l} c(I_i) + c(K)\epsilon/2\big(c(K)+1\big) \\
&< \epsilon
\end{aligned}$$

This implies that f is integrable on K.

Theorems: (1) Suppose f and g are integrable on a closed $n-$cell K, and $f = g$ everywhere on K except at a content zero subset Z of K, then $\int_K f = \int_K g$.

Proof. Since f, g are integrable on K and Z has content zero, $f-g$ is integrable on K and it is continuous with value 0 on $K \setminus Z$. Given $\epsilon > 0$, let $\{I_i\}_{i=1}^l$ be a collection of cells such that

$$Z \subset \mathrm{Int}\big(\bigcup_{i=1}^{l} I_i\big) \text{ and } \sum_{i=1}^{l} c(I_i) < \epsilon/4\big(\sup_K |f-g| + 1\big).$$

Let $P_\epsilon = \{K_j\}_{j=1}^m$ such that $\{I_i \cap K\}_{i=1}^l \subset P_\epsilon$. Thus, if P is any refinement of P_ϵ then

$$\begin{aligned}
|U_P(f-g) - L_P(f-g)| &\le |U_P(f-g, K\setminus \mathrm{Int}\big(\bigcup_{i=1}^{l} I_i\big)) - L_P(f-g, K\setminus \mathrm{Int}\big(\bigcup_{i=1}^{l} I_i\big))| \\
&+ \sum_{i=1}^{l} \big(\sup_{Z\cap I_i}(f-g) - \inf_{Z\cap I_i}(f-g)\big)c(I_i) \\
&< \epsilon.
\end{aligned}$$

This implies that $\int_K (f-g) = 0$. Since $\int_K g \in \mathbb{R}$, we have $\int_K f = \int_K g$.

(2) **Fundamental Theorem of Calculus** Let f be integrable on $[a,b]$. For each $x \in [a,b]$, let $F(x) = \int_{a,x]} f = \int_a^x f(t)dt$. Then F is continuous on $[a,b]$; moreover, $F'(x)$ exists and equals $f(x)$ at every x at which f is continuous.

Remark. For each $x \in [a,b]$, the existence of $F(x)$ is due to $[a,x] \subseteq [a,b]$ and the existence of $\int_a^b f$. Existence of $\int_a^b f$ implies that for each $\epsilon > 0$ $\exists$ a partition P_ϵ of $[a,b]$ such that if P is any refinement of P_ϵ, then $|U_P(f,[a,b]) - L_P(f,[a,b])| < \epsilon$.

Let

$$P_\epsilon^l = P_\epsilon \cap [a,x] \text{ and } P_\epsilon^r = P_\epsilon \cap [x,b].$$

Then $P_\epsilon^l \cup P_\epsilon^r$ is a refinement of P_ϵ and if P^l is any refinement of P_ϵ^l, then

$$\begin{aligned}
|U_{P_\epsilon^l}(f,[a,x]) - L_{P_\epsilon^l}(f,[a,x])| &\le |U_{P_\epsilon^l}(f,[a,x]) - L_{P_\epsilon^l}(f,[a,x]) + U_{P_\epsilon^r}(f,[x,b]) - L_{P_\epsilon^r}(f,[x,b])| \\
&= |U_{P_\epsilon^l\cup P_\epsilon^r}(f,[a,b]) - L_{P_\epsilon^l\cup P_\epsilon^r}(f,[a,b])| \\
&< \epsilon.
\end{aligned}$$

Hence, f is integrable on $[a,x]$, i.e. $F(x)$ exists.

Proof of the Theorem. If $x, y \in [a,b] \Rightarrow F(y) - F(x) = \int_x^y f(t)dt$.

Let $c = \sup\{|f(t)| : t \in [a,b]\}$. ($c$ exists since f is integrable on $[a,b] \Rightarrow f$ is bounded on $[a,b]$.)

Then $|F(y) - F(x)| \le |\int_x^y |f(t)|dt| \le c|\int_x^y dt| = c|y-x|$

$\Rightarrow F$ is (Lipschitz, hence) continuous on $[a,b]$.

Suppose that f is continuous at x; thus $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ such that if $t \in [a,b]$ and $|t-x| < \delta$ then $|f(t) - f(x)| < \epsilon$.

Since $f(x) = f(x)\,\frac{1}{y-x}\int_x^y dt = \frac{1}{y-x}\int_x^y f(x)dt$.
Hence, if $y \in [a,b]$ and $|y-x| < \delta \Rightarrow t \in [a,b]$ and $|t-x| < \delta$ for all t between y and x.
$\Rightarrow |f(t) - f(x)| < \epsilon$ and this implies that
$\Rightarrow |\dfrac{F(y)-F(x)}{y-x} - f(x)| \le \dfrac{1}{|y-x|}\,|\int_x^y |f(t)-f(x)|dt| \le \dfrac{1}{|y-x|}\,|\int_x^y \epsilon dt| = \epsilon.$
$\Rightarrow \lim\limits_{y\to x}\dfrac{F(y)-F(x)}{y-x} = f(x)$, i.e. $F'(x)$ exists and equals $f(x)$ at every x at which f is continuous.
(3) Let F be a continuous function on $[a,b]$ that is differentiable except at finitely many points in $[a,b]$, and let f be a function on $[a,b]$ that agrees with F' at all points where F' is defined. If f is integrable on $[a,b]$, then $\int_a^b f(t)dt = F(b) - F(a)$.
Proof. Let $\{z_1,\dots,z_m\} \subset [a,b]$ be the set at which F' does not exist, i.e. F is not differentiable at z_i, $i = 1,\dots,m$.
Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a,b]$ with z_i, $i = 1,\dots,m$, being partition point, i.e. each $z_i \in \{x_0, x_1, \dots, x_n\}$.
$\Rightarrow$ F is continuous on each $[x_{j-1}, x_j]$, $j = 1,\dots,n$, and differentiable on each (x_{j-1}, x_j).
By the Mean Value Theorem, $F(x_j) - F(x_{j-1}) = F'(t_j)\,(x_j - x_{j-1}) = f(t_j)\,(x_j - x_{j-1})$ for some $t_j \in (x_{j-1}, x_j)$ and for each $j = 1,\dots,n$.
Thus, we have $F(b) - F(a) = \sum\limits_{j=1}^{n} F(x_j) - F(x_{j-1}) = \sum\limits_{j=1}^{n} f(t_j)\,(x_j - x_{j-1})$
$\Rightarrow L_P(f,[a,b]) \le F(b) - F(a) \le U_P(f,[a,b])$
$\Rightarrow \sup\limits_P L_P(f,[a,b]) \le F(b) - F(a) \le \inf\limits_P U_P(f,[a,b])$
If f is integrable then $\int_a^b f(t)dt = \sup_P L_P(f,[a,b]) = \inf_P U_P(f,[a,b]) = F(b) - F(a)$.

Part 2: Integrable Functions on General Measurable Sets:

In the following we shall extend the concept of **content of a cell in** $\mathbb{R}^n$ to more general **measurable subsets of** $\mathbb{R}^n$ and extend the definition of integrability of a function to general subsets of $\mathbb{R}^n$.
Definition (of integrability on general Euclidaen bounded subsets.) Let $A \subset \mathbb{R}^n$ be a bounded set and let $f : A \to \mathbb{R}$ be a bounded function. Let K be a closed cell containing A and define $f_K : K \to \mathbb{R}$ by

$$f_K(x) = \begin{cases} f(x) & \text{if } x \in A, \\ 0 & \text{if } x \in K \setminus A. \end{cases}$$

We say that f is **integrable on** A **if** f_K **is integrable on** K, **and define** $\int_A f = \int_K f_K$.
Remark. If $A = K$ is a closed cell in $\mathbb{R}^n$, then, since $f_K = f$ on K, it is obvious the integrability of f on A agrees with the integrability of f_K on K and $\int_A f$ is defined to be $\int_K f_K$.
Remark. Let I be any closed cell containing A. Then $K \cap I$ is a closed cell containing A $\Rightarrow$ $f_K = f_{K\cap I} = f_I$ everywhere in $K \cap I$, $f_K = 0$ on $K \setminus (K \cap I)$, and $f_I = 0$ on $I \setminus (K \cap I)$. Hence, $\int_K f_K = \int_{K\cap I} f_{K\cap I} = \int_I f_I \Rightarrow$ the definition (of integrability of f) only depends on f and A (and it is independent of the choice of $K \supseteq A$).
Basic properties of integrable functions on general sets:
(1). Let f and g be integrable functions defined on a bounded set $A \subset \mathbb{R}^n$ and let $\alpha, \beta \in \mathbb{R}$. Then the function $\alpha f + \beta g$ is integrable on A and $\int_A (\alpha f + \beta g) = \alpha \int_A f + \beta \int_A g$.
Proof. For any partition P of a cell $K \supseteq A$, since $S_P(\alpha f_K + \beta g_K, K) = \alpha S_P(f_K, K) + \beta S_P(g_K, K)$ when the same intermediate points x_j are used, the function $\alpha f + \beta g$ is integrable on A. Thus, by choosing the intermediate points from A whenever it is possible, we obtain that $S_P(\alpha f + \beta g, A) = \alpha S_P(f, A) + \beta S_P(g, A)$ which implies that $\int_A (\alpha f + \beta g) = \alpha \int_A f + \beta \int_A g$.
(2) Let A_1 and A_2 be bounded sets with no pints in common, and let f be a bounded function. If f is integrable on A_1 and on A_2, then f is integrable on $A_1 \cup A_2$ and $\int_{A_1\cup A_2} f = \int_{A_1} f + \int_{A_2} f$.

Proof. Let K be a closed cell containing both A_1 and A_2, and let $f_K(x) = \begin{cases} f(x) & \text{if } x \in A_1 \cup A_2, \\ 0 & \text{if } x \in K \setminus (A_1 \cup A_2) \end{cases}$ and $f_K^i(x) = \begin{cases} f(x) & \text{if } x \in A_i, \\ 0 & \text{if } x \in K \setminus A_i \end{cases}$ for $i = 1, 2$. Since f is integrable on A_i, $i = 1, 2$, f_K^i is integrable on K and, since $f_K = f_K^1 + f_K^2$, and f is integrable on $A_1 \cup A_2$. Also, for any partition P of K, note that $S_P(f_K, K) = S_P(f_K^1, K) + S_P(f_K^2, K)$ when the same intermediate points x_j are used. Thus, we have $\int_{A_1 \cup A_2} f = \int_{A_1} f + \int_{A_2} f$.

(3) If $f : A \to \mathbb{R}$ is integrable on (bounded set) A and $f(x) \geq 0$ for $x \in A$, then $\int_A f \geq 0$.

Proof. For any closed cell $K \supseteq A$ and any partition P of K, note that $S_P(f_K, K) \geq 0$ for any Riemann sum. Thus, $\int_A f \geq 0$.

Remark. This implies that if f and g are integrable on A and $f(x) \leq g(x)$ for $x \in A$, then (a) $\int_A f \leq \int_A g$, and (b) $|f|$ is integable on A, and $|\int_A f| \leq \int_A |f|$.

(4) Let $f : A \to \mathbb{R}$ be a bounded function and suppose that A has content zero. Then f is integrable on A and $\int_A f = 0$.

Proof. Let$K \supseteq A$ be a closed cell. If $\epsilon > 0$ is given, let P_ϵ be a partition of K such that those cells in P_ϵ which contain points of A have total content less than ϵ. Now if P is a refinement of P_ϵ, then those cells in P containing points of A will also have total content less than ϵ. Hence if $|f(x)| \leq M$ for $x \in A$, we have $|S_P(f_K, K) - 0| \leq M\epsilon$ for any Riemann sum corresponding to P. Since ϵ is arbitrary, this implies that $\int_A f = 0$.

(5) Let $f, g : A \to \mathbb{R}$ be bounded functions and suppose that f is integrable on (bounded set) A. Let $Z \subseteq A$ have content zero and suppose that $f(x) = g(x)$ for all $x \in A \setminus Z$. Then g is integrable on A and $\int_A f = \int_A g$.

Proof. Extend f and g to functions f_K, g_K defined on a closed cell $K \supseteq A$. Thus, the function $h_K = f_K - g_K$ is bounded and equals 0 except on Z. Hence, h_K is integrable on K and $\int_K h_K = 0$. Since f_K is also integrable on K, we have $\int_A f = \int_K f_K = \int_K (f_K - h_K) = \int_K g_K = \int_A g$.

(6) Let U be a connected, open subset of $\mathbb{R}^n$ and let $f : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a C^1 map on U. If K is any convex, compact subset of U, then $f(K)$ has measure (or content) zero.

Definition. Let $A \subset \mathbb{R}^n$ be a bounded set. The **characteristic function** of A is the function χ_A defined by $\chi_A(x) = \begin{cases} 1 & \text{if} x \in A, \\ 0 & \text{otherwise.} \end{cases}$ Now, suppose A is a bounded subset of $\mathbb{R}^n$ and f is a bounded function on $\mathbb{R}^n$. Let K be a closed cell that contains A. We say that f is **integrable on** A if $f\chi_A$ is integrable on K, and define $\int_A f = \int_K f\chi_A$. (Note that $f\chi_A = 0$ on $K \setminus A$, so it is independent of the choice of $K \supseteq A$.)

Question: Let $f \equiv 1$ on $A \subset \mathbb{R}^n$. What does it mean when we say that f is integrable on A?

Definition. A set $A \subset \mathbb{R}^n$ is said to have content (or it is said to be (Jordan) measurable) if it is bounded and its boundary ∂A has content zero. Let $\mathscr{D}(\mathbb{R}^n)$ $(= \{A \subset \mathbb{R}^n \mid A \,\text{has content}\} = \{A \subset \mathbb{R}^n \mid A \,\text{is measurable}\})$ denote the set of all measurable subsets of $\mathbb{R}^n$.

Remark. If $A \in \mathscr{D}(\mathbb{R}^n)$ and if K is a closed cell containing A, then the function g_K defined by $g_K(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in K \setminus A \end{cases}$ is continuous on K except possibly at points of ∂A (which has content zero). Thus, g_K is integrable on K and we define the content of A, denoted $c(A)$, by $c(A) = \int_K g_K = \int_A 1$.

Remark. $\forall\, \epsilon > 0$, since $c(A) = \int_K g_K$, $\exists$ a partition $P_\epsilon = \{I_j\}_{j=1}^m$ of K such that $|S_{P_\epsilon}(g_k, K) - c(A)| < \epsilon$ for any Riemann sum $S_{P_\epsilon}(g_k, K)$. By choosing the intermediate points in $S_{P_\epsilon}(g_k, K)$ to belong to A whenever possible, we have $\sum_{j=1}^m c(I_j) + \epsilon \geq S_{P_\epsilon}(g_k, K) + \epsilon > c(A) > S_{P_\epsilon}(g_k, K) - \epsilon$, where the first inequality holds since $A \subset \cup_{j=1}^m I_j$.

Thus, we have: **A set $A \subset \mathbb{R}^n$ has content zero if and only if A has content and** $\int_A 1 = c(A) = 0$.

Proof. ($\Rightarrow$) $\forall\, \epsilon > 0$, since A has content zero, $\exists$ closed cells $I_1, \ldots, I_m$ s.t. $\begin{cases} A \subset \cup_{j=1}^m I_j = K_\epsilon \\ \sum_{j=1}^m c(I_j) < \epsilon. \end{cases}$
Since (i) K_ϵ is bounded $\Rightarrow$ A is bounded, and (ii) K_ϵ is closed $\Rightarrow$ $\partial A \subset K_\epsilon = \cup_{j=1}^m I_j$ with $\sum_{j=1}^m c(I_j) < \epsilon \Rightarrow c(\partial A) = 0$
$\Rightarrow$ A has content and $c(A) = \int_A 1 = \int_K g_K = 0$ since $0 \le c(A) < S_{P_\epsilon}(g_K, K) + \epsilon \le \sum_{j=1}^m c(I_j) + \epsilon \le 2\epsilon$ and ϵ is arbitrary.
($\Leftarrow$) Suppose that $A \subset \mathbb{R}^n$ has content and that $c(A) = 0 \Rightarrow \exists$ a closed cell K containing A s.t. the function $g_K(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in K \setminus A \end{cases}$ is integrable on K. $\forall\, \epsilon > 0$, let $P_\epsilon = \{I_j\}_{j=1}^m$ be a partition of K s.t. any Riemann sum corresponding to P_ϵ satisfies that $0 \le |S_{P_\epsilon}(g_K, K) - c(A)| < \epsilon$. Since $c(A) = 0$, we have $0 \le S_{P_\epsilon}(g_K, K) < \epsilon$. By taking the intermediate points in $S_{P_\epsilon}(g_K, K)$ to belong to A whenever possible, we have $A \subset \displaystyle\bigcup_{1 \le j \le m;\ I_j \cap A \neq \emptyset} I_j$ and $\displaystyle\sum_{1 \le j \le m;\ I_j \cap A \neq \emptyset} c(I_j) < \epsilon \Rightarrow c(A) = 0$.
Theorem. Let $A \in \mathscr{D}(\mathbb{R}^n)$ and let $f : A \to \mathbb{R}$ be integrable on A and such that $|f(x)| \le M$ for all $x \in A$. Then $|\int_A f| \le Mc(A)$. More generally, if f is real valued and $m \le f(x) \le M$ for all $x \in A$, then $(*)$ $mc(A) \le \int_A f \le Mc(A)$.
Proof. Let f_K be the extension of f to a closed cell K containing A. If $\epsilon > 0$ is given, then there exists a partition $P_\epsilon = \{I_j\}_{j=1}^h$ of K such that if $S_{P_\epsilon}(f_K, K)$ is any corresponding Riemann sum, then $S_{P_\epsilon}(f_K, K) - \epsilon \le \int_K f_K \le S_{P_\epsilon}(f_K, K) + \epsilon$. We note that if the intermediate points of the Riemann sum are chosen outside of A whenever possible, we have $S_{P_\epsilon}(f_K, K) = \sum' f(x_j)c(I_j)$, where the sum is extended over those cells in P_ϵ entirely contained in A. Hence, $S_{P_\epsilon}(f_K, K) \le M \sum' c(I_j) \le Mc(A)$. Therefore, we have $\int_A f = \int_K f_K \le Mc(A) + \epsilon$, and since $\epsilon > 0$ is arbitrary we obtain the right side of inequality $(*)$. The left side is established in a similar manner.
Theorem. If $A \in \mathscr{D}(\mathbb{R}^n)$ and $c(A) > 0$, then there exists a closed cell $J \subseteq A$ such that $c(J) \neq 0$.
Mean Value Theorem. Let $A \in \mathscr{D}(\mathbb{R}^n)$ be a connected set and let $f : A \to \mathbb{R}$ be bounded and continuous on A. Then there exists a point $p \in A$ such that $\int_A f = f(p)c(A)$.
Proof. If $c(A) = 0$, the conclusion is trivial; hence we suppose that $c(A) \neq 0$. Let $m = \inf\{f(x) : x \in A\}$, and $M = \sup\{f(x) : x \in A\}$; it follows from the preceding theorem that $m \le \frac{1}{c(A)} \int_A f \le M$. If both inequalities are strict, the results follows from Intermediate Value Theorem (since f is continuous on A). Now suppose that $\int_A f = Mc(A)$. If the supremum M is attained at $p \in A$, the conclusion also follows. Hence we assume that the supremum M is not attained on A. Since $c(A) \neq 0$, there exists a closed cell $J \subseteq A$ such that $c(J) \neq 0$ (prove this). Since J is compact and f is continuous on J, there exists $\epsilon > 0$ such that $f(x) \le M - \epsilon$ for all $x \in J$. Since $A = J \cup (A \setminus J)$ we have $Mc(A) = \int_A f = \int_J f + \int_{A \setminus J} f \le (M - \epsilon)c(J) + Mc(A \setminus J) < Mc(A)$, a contradiction. If $\int_A f = mc(A)$, then a similar argument applies.
Mean Value Theorem for Riemann-Stieltjes Integrals. Let $A \in \mathscr{D}(\mathbb{R}^n)$ be a connected set, let $f : A \to \mathbb{R}$ be bounded and continuous on A, and let $g : A \to \mathbb{R}$ be bounded, nonnegative and continuous on A, Then there exists a point $p \in A$ such that $\int_A fg = f(p) \int_A g$.
Remark (Well-definedness of (Jordan) measurability) . Note that the definition for a set having content (or the definition of a set being measurable) is well defined since the the following properties hold.
Proposition. Let A, $B \in \mathscr{D}(\mathbb{R}^n)$ and let $x \in \mathbb{R}^n$. Then
(1) $A \cap B$, $A \cup B \in \mathscr{D}(\mathbb{R}^n)$ and $c(A) + C(B) = c(A \cap B) + c(A \cup B)$. Using induction, this concludes that if $A \in \mathscr{D}(\mathbb{R}^n)$ and A is a finite disjoint union of measurable subsets, i.e. $A = \displaystyle\bigcup_{i=1}^{l} B_i$ and each $B_i \in \mathscr{D}(\mathbb{R}^n)$, then $c(A) = \displaystyle\sum_{i=1}^{l} c(B_i)$.
(2) $A \setminus B$, $B \setminus A \in \mathscr{D}(\mathbb{R}^n)$, and $c(A \cup B) = c(A \setminus B) + c(A \cap B) + c(B \setminus A)$.

(3) If $x+A = \{x+a \mid a \in A\}$ then $x+A \in \mathscr{D}(\mathbb{R}^n)$ and $c(x+A) = c(A)$, i.e. the definition of content is invariant under translations.

Proof. By using the definition of boundary points of a set, we have $\partial(A \cap B)$, $\partial(A \cup B)$, $\partial(A \setminus B)$, $\partial(B \setminus A) \subset \partial A \cup \partial B$. Thus, $c(\partial(A \cap B)) = c(\partial(A \cup B)) = c(\partial(A \setminus B)) = c(\partial(B \setminus A)) = 0$ and $A \cap B$, $A \cup B$, $A \setminus B$, $B \setminus A \in \mathscr{D}(\mathbb{R}^n)$.

Now let $K \supseteq A \cup B$ be a closed cell and let $f_A, f_B, f_{A\cap B}, f_{A\cup B}$ be the functions equal to 1 on $A, B, A \cap B, A \cup B$, respectively, and equal 0 elsewhere on K. Then they are integrable on K and , since $f_A + f_B = f_{A\cap B} + f_{A\cup B}$, we have

$c(A) + c(B) = \displaystyle\int_K f_A + \int_K f_B = \int_K (f_A + f_B) = \int_K (f_{A\cap B} + f_{A\cup B}) = \int_K f_{A\cap B} + \int_K f_{A\cup B}$

$= c(A \cap B) + c(A \cup B)$.

To prove (3), note that if $\epsilon > 0$ is given and if $J_1, \ldots, J_m$ are cells with $\partial A \subset \displaystyle\bigcup_{i=1}^{m} J_i$ and $\displaystyle\sum_{i=1}^{m} c(J_i) < \epsilon$, then $x + J_1, \ldots, x + J_m$ are cells with $\partial(x+A) \subset \displaystyle\bigcup_{i=1}^{m}(x + J_i)$ and $\displaystyle\sum_{i=1}^{m} c(x + J_i) < \epsilon$. Since $\epsilon > 0$ is arbitrary, the set $x + A$ belongs to $\mathscr{D}(\mathbb{R}^n)$.

Let I be a closed cell containing A; hence $x + I$ is a closed cell containing $x + A$. Let $f_1 : I \to \mathbb{R}$ be such that $f_1(y) = 1$ for $y \in A$ and $f_1(y) = 0$ for $y \in I \setminus A$, and let $f_2 : x + I \to \mathbb{R}$ be such that $f_2(y) = 1$for $y \in x + A$ and $f_1(y) = 0$ for $y \in x + I \setminus (x + A)$. Thus, we have $f_1(y) = f_2(x + y)$ for each $y \in A$, and $c(A) = \int_I f_1 = \int_{x+I} f_2 = c(x + A)$.